Galois scaffolds and semistable extensions

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References

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Local Fields

Let K be a field which is complete with respect to a discrete valuation $v_K : K^\times \to \mathbb{Z}$, whose residue field $\overline{K}$ is a perfect field of characteristic p. Also let

$$\begin{aligned}
\mathcal{O}_K &= \{\alpha \in K : v_K(\alpha) \geq 0\} \\
&= \text{ring of integers of } K \\
\\
\pi_K &= \text{uniformizer for } \mathcal{O}_K \text{ (i.e., } v_K(\pi_K) = 1) \\
\\
\mathcal{M}_K &= \pi_K \mathcal{O}_K \\
&= \text{unique maximal ideal of } \mathcal{O}_K
\end{aligned}$$

Let L/K be a totally ramified Galois extension of degree $q = p^n$, and set $G = \mathrm{Gal}(L/K)$.

Galois scaffolds (setup)

Let $b_1 \leq b_2 \leq \cdots \leq b_n$ be the lower ramification breaks of L/K, counted with multiplicity. Assume that $p \nmid b_i$ for $1 \leq i \leq n$. Set $\mathbb{S}_{p^n} = \{0, 1, \ldots, p^n - 1\}$ and write $s \in \mathbb{S}_{p^n}$ in base p as

$$s = s_{(0)}p^0 + s_{(1)}p^1 + \cdots + s_{(n-1)}p^{n-1}$$

with $0 \leq s_{(i)} < p$. Define $\mathfrak{b} : \mathbb{S}_{p^n} \to \mathbb{Z}$ by

$$\mathfrak{b}(s) = s_{(0)}p^0 b_n + s_{(1)}p^1 b_{n-1} + \cdots + s_{(n-1)}p^{n-1} b_1.$$

Let $r : \mathbb{Z} \to \mathbb{S}_{p^n}$ be the function which maps $a \in \mathbb{Z}$ onto its least nonnegative residue modulo p^n. The function $r \circ (-\mathfrak{b}) : \mathbb{S}_{p^n} \to \mathbb{S}_{p^n}$ is a bijection since $p \nmid b_i$. Therefore we may define $\mathfrak{a} : \mathbb{S}_{p^n} \to \mathbb{S}_{p^n}$ to be the inverse of $r \circ (-\mathfrak{b})$. Extend $\mathfrak{a}$ to a function from $\mathbb{Z}$ to $\mathbb{S}_{p^n}$ by setting $\mathfrak{a}(t) = \mathfrak{a}(r(t))$ for $t \in \mathbb{Z}$.

Galois scaffolds

Definition (cf. [BCE], Definition 2.6)

A Galois scaffold for L/K consists of elements $\Psi_i \in K[G]$ for $1 \leq i \leq n$ such that the following hold:

1. $\Psi_i(1) = 0$ for $1 \leq i \leq n$.
2. Let $\lambda \in L^\times$ and set $t = v_L(\lambda)$. Then for $1 \leq i \leq n$ we have

$$v_L(\Psi_i(\lambda)) = t + p^{n-i} b_i \text{ if } \mathfrak{a}(t)_{(n-i)} \geq 1,$$
$$v_L(\Psi_i(\lambda)) > t + p^{n-i} b_i \text{ if } \mathfrak{a}(t)_{(n-i)} = 0.$$

A Galois scaffold for L/K can be used to get information about the Galois module structure of ideals in $\mathcal{O}_L$. It can also be used to give sufficient conditions for the associated order of $\mathcal{O}_L$ to be a Hopf order.

The Map $\phi : L \otimes_K L \to L[G]$

There is a K-linear map $\phi : L \otimes_K L \to L[G]$ defined by

$$\phi(a \otimes b) = \sum_{\sigma \in G} a\sigma(b)\sigma.$$

For $x \in L$ we get

$$\phi(a \otimes b)(x) = \sum_{\sigma \in G} a\sigma(bx) = a\mathrm{Tr}_{L/K}(bx).$$

Proposition ([Bon1], Proposition 1.1.2)

ϕ is an isomorphism of K-vector spaces.

A Partial Order

Recall that $[L : K] = p^n = q$.

Let H be the subgroup of $\mathbb{Z} \times \mathbb{Z}$ generated by the element $(q, -q)$.

For $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ write $[a, b]$ for the coset $(a, b) + H$.

We define a partial order on the quotient group $(\mathbb{Z} \times \mathbb{Z})/H$ by $[a, b] \preceq [c, d]$ if and only if there is $(c', d') \in [c, d]$ such that $a \leq c'$ and $b \leq d'$.

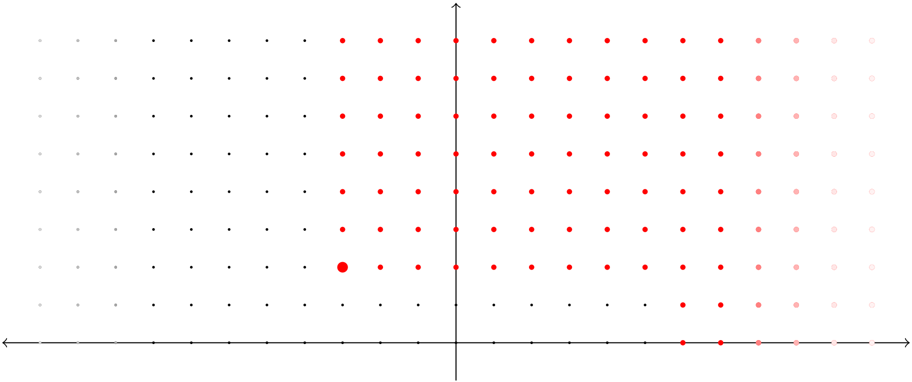
We often use the following set of coset representatives for $(\mathbb{Z} \times \mathbb{Z})/H$:

$$\mathcal{F} = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : 0 \leq b < q\}$$

An Example

Let $q = 9$. Here is the set

$$\{(c, d) \in \mathcal{F} : [-3, 2] \preceq [c, d]\}.$$



Expansions of Tensors

Choose a uniformizer π_L for L and let $\mathcal{T}$ be the set of Teichmüller representatives of K.

Let $\beta \in L \otimes_K L$. Then there are unique $a_{ij} \in \mathcal{T}$ such that

$$\beta = \sum_{(i,j)\in\mathcal{F}} a_{ij}\pi_L^i \otimes \pi_L^j.$$

Set

$$R(\beta) = \{[i,j] : (i,j) \in \mathcal{F},\ a_{ij} \neq 0\}.$$

Then $R(\beta)$ depends on the choice of π_L.

Diagrams and Diagonals

Definition

Define the diagram of $\beta \in L \otimes_K L$ to be

$$D(\beta) = \{[x, y] \in (\mathbb{Z} \times \mathbb{Z})/H : [i, j] \preceq [x, y] \text{ for some } [i, j] \in R(\beta)\}.$$

Proposition ([Bon2], Remark 2.4.3)

$D(\beta)$ does not depend on the choice of uniformizer π_L for L.

For $\beta \in L \otimes_K L$ with $\beta \neq 0$ define

$$d(\beta) = \min\{i + j : [i, j] \in D(\beta)\}.$$

Define the diagonal of β to be

$$N(\beta) = \{[i, j] \in D(\beta) : i + j = d(\beta)\}.$$

The Generating Set of a Diagram

Let $G(\beta)$ denote the set of minimal elements of $D(\beta)$. Then $N(\beta) \subset G(\beta)$.

Set $i_0 = \mathfrak{b}(p^n - 1)$. Then $i_0 + p^n - 1 = v_L(\delta_{L/K})$ is the valuation of the different of L/K.

Theorem ([Bon2], Proposition 2.4.2)

Let $\beta \in L \otimes_K L$ be such that $\xi := \phi(\beta) \in K[G]$. Then the following statements are equivalent:

1. $[a, b] \in G(\beta)$.
2. *For all $y \in L$ with $v_L(y) = -b - i_0$ we have $v_L(\xi(y)) = a$.*

An Example

Let $q = 9$ and set

$$\beta = a_{50}\pi_L^5 \otimes \pi_L^0 + a_{44}\pi_L^4 \otimes \pi_L^4 + a_{24}\pi_L^2 \otimes \pi_L^4 + a_{05}\pi_L^0 \otimes \pi_L^5.$$

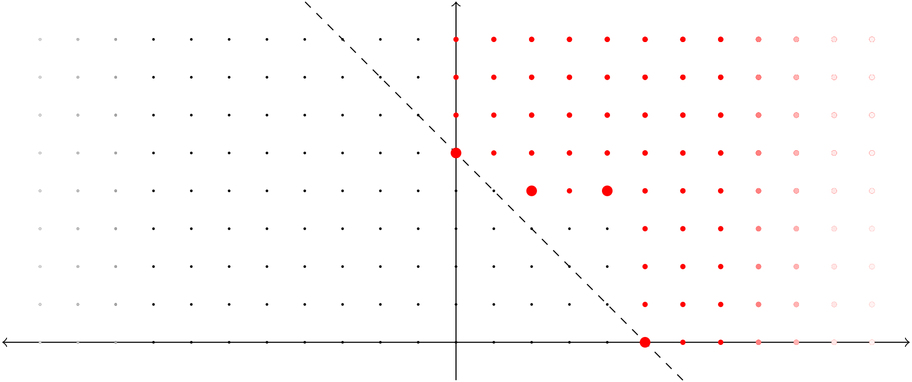
with $a_{ij} \in \mathcal{T} \smallsetminus \{0\}$. We get

$$\begin{aligned} R(\beta) &= \{[5,0],[4,4],[2,4],[0,5]\} \\ G(\beta) &= \{[5,0],[2,4],[0,5]\} \\ N(\beta) &= \{[5,0],[0,5]\} \\ d(\beta) &= 5. \end{aligned}$$

The subset of $\mathcal{F}$ corresponding to $D(\beta)$ is ...

Example Diagram

$$q = 9, \quad \beta = a_{50}\pi_L^5 \otimes \pi_L^0 + a_{44}\pi_L^4 \otimes \pi_L^4 + a_{24}\pi_L^2 \otimes \pi_L^4 + a_{05}\pi_L^0 \otimes \pi_L^5$$



Semistable Extensions

Definition

Say that the extension L/K is semistable if there is $\beta \in L \otimes_K L$ such that $\phi(\beta) \in K[G]$, $p \nmid d(\beta)$, and $|N(\beta)| = 2$.

The element $\phi(\beta)$ can be used to get information about the Galois module structure of ideals in $\mathcal{O}_L$. It can also be used to give sufficient conditions for the associated order of $\mathcal{O}_L$ to be a Hopf order.

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Theorem

Let L/K be a totally ramified Galois extension of degree $q = p^n$. Then L/K is semistable if and only if L/K has a Galois scaffold.

Galois Scaffold $\Rightarrow$ Semistable

Suppose $\{\Psi_1, \ldots, \Psi_n\}$ is a Galois scaffold for L/K. Set

$$\xi = \Psi^{(p^n-2)} = \Psi_n^{p-2}\Psi_{n-1}^{p-1}\ldots\Psi_2^{p-1}\Psi_1^{p-1}.$$

For $y \in L^\times$ we get

$$\begin{aligned}
&v_L(\xi(y)) = v_L(y) + \mathfrak{b}(p^n-2) \text{ if } v_L(y) \equiv \mathfrak{b}(p^n-1) \pmod{p^n} \\
&v_L(\xi(y)) = v_L(y) + \mathfrak{b}(p^n-2) \text{ if } v_L(y) \equiv \mathfrak{b}(p^n-2) \pmod{p^n} \\
&v_L(\xi(y)) > v_L(y) + \mathfrak{b}(p^n-2) \text{ otherwise.}
\end{aligned}$$

Let $\beta \in L \otimes_K L$ be such that $\phi(\beta) = \xi$. It follows from the above that

$$N(\beta) = \{[-b_n, 0], [0, -b_n]\}.$$

Therefore L/K is semistable.

Semistable $\Rightarrow$ Galois Scaffold

There is $\xi \in K[G]$ and $\beta \in L \otimes_K L$ such that $\phi(\beta) = \xi$ and $|N(\beta)| = 2$.

By [Bon2, Prop. 3.2] we can assume $N(\beta) = \{[i_0, 0], [0, i_0]\}$.

For $1 \leq i \leq n$ define $\Theta_i = \phi(\beta^{p^n - p^{n-i} - 1})$. Then $\Theta_i \in K[G]$, and there is c_i such that $c_i \equiv p^{n-i} b_i \pmod{p^n}$ with the following property:

Let $\lambda \in L^\times$ and set $t = v_L(\lambda)$. Then $v_L(\Theta_i(t)) \geq t + c_i$, with equality if and only if $\mathfrak{a}(t)_{(n-i)} \geq 1$.

Set $v_i = (c_i - p^{n-i} b_i)/p^n$. Then $\Psi_i = \pi_K^{-v_i} \Theta_i$ satisfies $v_L(\Psi_i(\lambda)) \geq t + p^{n-i} b_i$, with equality if and only if $\mathfrak{a}(t)_{(n-i)} \geq 1$.

Hence $\{\Psi_1, \ldots, \Psi_n\}$ is a Galois scaffold for L/K.